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Internal-time observable of classical relativistic systems

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Abstract

The relativistic framework with its symmetries offers a natural definition for the internal time of classical (non-quantum) physical systems as a Lorentz-invariant observable. The internal-time observable, measuring the system's aging or internal evolution, is identified with the proper time of the system derived from its centre-of-mass (CM) coordinate. For its definition as an observable it is required that the system be symmetric not only under Lorentz–Poincaré transformations but also under uniform scaling, with the associated existence of a dilatation function D, and yet that D be a varying—not conserved—quantity. Two alternative definitions are discussed, and it is found that in order to maintain simultaneity of the CM time with the events that define it, it is necessary to split the dilatation function into a CM part and an internal part.

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1. Introduction

The purpose of this paper is to derive and discuss, in a Lorentz-covariant manner, the concept of *internal time*, its existence and properties, for classical (i.e., non-quantum) relativistic systems of interacting particles, as an observable which is dependent on and defined by the particles' degrees of freedom.

The motivation for looking for an internal-time observable comes from the simple fact that complex systems evolve in time, their evolution embedded in their internal dynamics. It is therefore reasonable and natural to expect the existence of an observable which is capable of measuring, in some sense, the time evolution or *aging* of the system. Such a time-like observable is *internal*, in the sense that it depends on the dynamical variables of the constituents of the system, and at the same time is *global* in the sense that it provides a property that corresponds to the system as a whole (in the same way, in principle, as are the total linear and angular momenta).

The concept of internal time of a physical system, defined from its own degrees of freedom and being a means to measure its aging, gained some attention in recent years. Internal time was introduced, in the context of non-equilibrium statistical mechanics, by Misra, Prigogine and Courbage [1] as an operator acting on distribution functions in phase space. A more general approach was proposed few years ago also by Vallèe (see [2] and references therein). However, common to all these approaches is that they are rooted in the Newtonian frame of mind, in which time is only external to all physical systems—an absolute, referential parameter, common to all. As such, Newtonian time cannot be dynamically related to the internal evolution of any physical system. Consequently, definitions of internal time in Newtonian systems can only be ad hoc.

But in relativistic dynamics time has a dual nature. On the one hand, the sense of time—as defined by the future light-cone—is external, transcendental, common to all; and in any inertial frame the observer's time, attached to that frame, serves as an external reference time for all physical systems. On the other hand, the *proper time* of particles and systems, which measures locally how time elapses for these particles and systems, is different and exclusive for each particle or system. Being Lorentz invariant, the proper time may be regarded as a measure of the *age* of the particle or system. This dichotomy of the relativistic time (which is well and elegantly illustrated by the so-called twin paradox) provides the golden path towards a natural definition of the internal-time observable.

For point particles lacking the internal structure the proper time is, and can only be, defined from their trajectory in spacetime relative to an external referential time. But for complex systems with an internal structure and evolution it is reasonable to expect that their proper time—as the measure of age—is also related to their internal degrees of freedom, so that it becomes a bridge between the internal structure of the system and the external time measurement process.

The proper time of complex systems is therefore expected to have a double definition: one, in relation to the external referential time, and another, in relation to the inner constitution of the system. The external definition must be associated with the centre-of-mass (CM) trajectory that describes the motion of the system as a whole in spacetime. The internal definition should be as follows. Any system of particles is described, classically, by the bundle of trajectories they traverse in spacetime. Picking arbitrarily an event x_a^μ on each particle's trajectory, to any such set of events there should correspond a proper-time value which is a functional of these events, $\tau[\{x_a\}]$, which may be interpreted as the corresponding age of the system as a whole. Certainly, to each value of the proper time there corresponds in this way a whole range of different such sets of events, not simultaneous in general, but there is no logical difficulty about this fact. The proper time defined in this way is a Lorentz-invariant observable, in accordance with the fact that the *age* of a system should be intrinsic and independent of any particular reference frame, and thus can be identified with the sought-for internal-time observable.

The plan of the paper is as follows. In section 2 we recall the generic form of the relativistic CM coordinate of systems with conserved linear and angular momenta. While its spatial aspects have been well studied for several decades already, a corresponding analysis of its time-like aspects is lacking. This is done in section 3, where it is found that application of Lorentz–Poincaré symmetries necessarily leads to the existence of a non-conserved dilatation function and two alternative definitions of the proper time in terms of it. Thus it turns out that these systems must also enjoy some kind of symmetry under dilatations. In section 4 we obtain explicit expressions of the dilatation function for some families of many-particle systems. A non-trivial issue of simultaneity between the CM-observer time and the proper-time observable then serves in section 5 to examine the two alternative definitions of the proper-time observable. It is shown in section 6 that in order to maintain plain simultaneity, in the standard sense, it is necessary to split the dilatation function into a CM part and an internal part. Then only the CM part participates in the definition of the proper-time observable, while the other provides a measure of the configuration of the spatial system and its deviation from full symmetry. Section 7 contains final discussion and concluding remarks.

Notation. In the following, we consider dynamics described in a Minkowski spacetime $\{x^\mu\}$, $\mu = 0, 1, 2, 3$, with metric tensor $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Space components only are indexed by Latin letters. The unit fully anti-symmetric (Levi-Civita) pseudo-tensor is $\varepsilon^{\mu\nu\lambda\rho} = -\varepsilon_{\mu\nu\lambda\rho} = 1$ for $(\mu\nu\lambda\rho)$ an even permutation of $(0, 1, 2, 3)$. It is also assumed throughout that $c = 1$ unless specified otherwise.

2. The relativistic centre-of-mass

As indicated in the introduction, the internal-time observable should be defined, as the proper time, from the CM trajectory that characterizes the motion of the system as a whole. Let us recall the relevant and main aspects of the relativistic centre-of-mass.

The search for a proper definition of centre-of-mass for relativistic systems stretches from the middle of the twentieth century [3]. While the Newtonian definition of the centre-of-mass is very simple and straightforward, it is the dual nature of time in relativistic dynamics that prevents simply carrying the Newtonian definition over to the relativistic realms.

In Newtonian dynamics the centre-of-mass can be defined either via the coordinates,

$$\boldsymbol{R}_{\mathrm{CM}} = \frac{\sum_a m_a \boldsymbol{r}_a}{M} \tag{1}$$

with $M = \sum_a m_a$, or as moving with the total momentum of the physical system,

$$\boldsymbol{V}_{\mathrm{CM}} = \frac{\boldsymbol{P}_{\mathrm{tot}}}{M} \tag{2}$$

These two definitions are compatible, via the relation $\boldsymbol{V}_{\mathrm{CM}} = \mathrm{d}\boldsymbol{R}_{\mathrm{CM}}/\mathrm{d}t$, because Newtonian time is universal and common to all. In relativistic dynamics, the compatibility is lost because individual particles' momenta are defined with respect to each particle's proper time. Thus the relativistic CM reference frame is usually defined in the second sense, as moving with the total momentum of the physical system, while a new relation replacing equation (1) is sought for.

While the CM velocity given by the relativistic counterpart of equation (2), $\boldsymbol{V}_{\mathrm{CM}} = \boldsymbol{P}_{\mathrm{tot}}/E_{\mathrm{tot}}$, requires only the knowledge of the total energy–momentum of the system, the rhs of equation (1) is related (Newtonianly) with the boost of Galilean transformations and thus requires relativistically also the knowledge of the total angular-momentum tensor, the generator of homogeneous Lorentz transformations. It is therefore assumed in the following that the systems considered are closed, endowed with conserved total linear momentum P^μ, which also defines the total invariant mass of the system $M = \sqrt{-P^\mu P_\mu}$, and conserved total angular momentum $J^{\mu\nu}$. P^μ and $J^{\mu\nu}$ are the fundamental global observables that describe the system, and at the same time are assumed to be the generators of global Lorentz–Poincaré transformations. It is then expected that the CM coordinate is constructed from P^μ and $J^{\mu\nu}$ together, possibly, with other observables that describe the system.

Since the CM frame moves, relative to a general inertial reference frame, with unit velocity vector

$$U^\mu = \frac{P^\mu}{M}$$

the trajectory of the CM coordinate in spacetime (denoted here by X^μ) may always be written as

$$X^\mu(\tau) = R^\mu + \tau \cdot U^\mu \tag{3}$$

where R^μ is a constant 4-vector and τ can clearly be regarded as the CM proper time. Appropriately fixing the zero of τ, R^μ may be assumed orthogonal to P^μ without loss of generality,

$$R^\mu P_\mu = 0. \tag{4}$$

R^μ may clearly be regarded as the constant spatial part of the CM coordinate. Unlike the Newtonian case (equation (1)) in which the CM coordinate is uniquely defined, in the relativistic case R^μ may be defined in an infinite number of ways, depending upon the requirements and specifications upon X^μ or, what turns out to be equivalent, upon the internal angular momentum: the internal angular momentum $J^{\mu\nu}_{\text{int}}$ (not necessarily a spin tensor!) is the result of splitting the total angular momentum into an orbital (CM) part and an internal part,

$$J^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu + J^{\mu\nu}_{\text{int}} = R^\mu P^\nu - R^\nu P^\mu + J^{\mu\nu}_{\text{int}} \tag{5}$$

Out of the six components of $J^{\mu\nu}_{\text{int}}$, three are determined by the condition

$$\epsilon_{\mu\nu\lambda\rho}\left(J^{\mu\nu} - J^{\mu\nu}_{\text{int}}\right)P^\lambda = 0$$

which follows from equation (5), leaving three independent degrees-of-freedom. These are incorporated in the generic expression for R^μ which follows from equations (4) and (5) as [4]

$$R^\mu = \frac{\left(J^{\mu\nu} - J^{\mu\nu}_{\text{int}}\right)P_\nu}{P^\nu P_\nu} = -\frac{\left(J^{\mu\nu} - J^{\mu\nu}_{\text{int}}\right)P_\nu}{M^2}.$$

Various approaches to the relativistic CM issue have yielded a variety of expressions for R^μ. Since these differences are immaterial for the purposes of the present paper they will not be discussed here. For our purposes, it suffices to know that R^μ is fully definable as a constant vector observable in terms of the dynamical variables of the constituents of the system.

So far, the above is well known. By this scheme, the CM coordinate is defined, except for τ, in terms of the dynamical variables of the constituents of the system. τ, on the other hand, was considered just as a time-like parameter and its definition as an observable remained unknown. Thus, we now turn to establishing the necessary dynamic relations of τ with the observables of the system, bringing τ to the level of an observable itself.

3. The internal-time observable

Let us start considering the symmetry properties of τ. The CM coordinate X^μ is expected to behave like an ordinary spacetime coordinate under the symmetries of the Minkowski spacetime, in particular like a 4-vector under homogeneous Lorentz transformations and be appropriately translated under uniform translations,

$$x^\mu \to x^\mu + a^\mu \quad \Rightarrow \quad X^\mu \to X^\mu + a^\mu. \tag{6}$$

Contracting equation (3) with P_μ yields

$$\tau = -\frac{X^\mu P_\mu}{M}. \tag{7}$$

With X^μ being a 4-vector in Minkowski spacetime, it follows that τ must be a Lorentz scalar. From the translational transformation (6) of X^μ it follows that under uniform translations τ transforms as

$$x^\mu \to x^\mu + a^\mu \quad \Rightarrow \quad \tau \to \tau - \frac{a^\mu P_\mu}{M}. \tag{8}$$

Since X^μ is expected to be an observable of the system, then so should τ be: the sensitivity of τ to translations and the fact that the addition to τ in (8) depends on P^μ (and is therefore an observable by itself) necessarily imply that τ *must* be an observable as well.

Consider now the product $X \cdot P = X^\mu P_\mu$. This should be a scalar observable, and under uniform translations it follows from (6) that $X^\mu P_\mu$ transforms as

$$x^\mu \to x^\mu + a^\mu \quad \Rightarrow \quad X^\mu P_\mu \to X^\mu P_\mu + a^\mu P_\mu. \tag{9}$$

It cannot be constructed out of P^μ and $J^{\mu\nu}$ only, because any quantity which is built from P^μ and $J^{\mu\nu}$ alone must be a constant while τ and thus also $X^\mu P_\mu$ must be a time-like variable, and in any case, it is impossible to construct from P^μ and $J^{\mu\nu}$ alone a scalar which satisfies equation (9). Thus another global observable, distinct from P^μ and $J^{\mu\nu}$, is required.

The global observable which has the same transformation properties as $X^\mu P_\mu$ is the dilatation function D, the generator of uniform scaling: all dilatation functions are Lorentz scalars and transform under uniform translations as

$$x^\mu \to x^\mu + a^\mu \quad \Rightarrow \quad D \to D + a^\mu P_\mu. \tag{10}$$

For a single free particle D is indeed given by $x^\mu p_\mu$; for a system of free particles it is given by a sum of the corresponding products, and interactions may also be included (see the next section). We may therefore proceed with firm knowledge that D is definable and exists. However, can we *identify* $X^\mu P_\mu$ with D?

The simplest, straightforward approach to the CM coordinate issue requires that X^μ be constructed only from the generators of the global Minkowski spacetime symmetries, as indeed is the case in Newtonian dynamics. Then $R^\mu = -J^{\mu\nu} P_\nu / M^2$ and $X^\mu P_\mu$ must be identified with D, and the proper-time observable is necessarily defined as

$$\tau \equiv -\frac{D}{M}. \tag{11}$$

On the other hand, if $X^\mu P_\mu \neq D$ then D may be regarded as being split, like $J^{\mu\nu}$ in equation (5), into a CM part ($X^\mu P_\mu$) and an internal part in the form

$$D = D_{\mathrm{CM}} + D_{\mathrm{int}} = X^\mu P_\mu + D_{\mathrm{int}} \tag{12}$$

In this case, the proper-time observable should be defined as

$$\tau \equiv -\frac{D - D_{\mathrm{int}}}{M}. \tag{13}$$

The next sections are dedicated to discussing and deciding upon these two alternatives. Concluding the present section, it is noted first that an inevitable result of the existence of the dilatation function is that the systems in question, in addition to being Lorentz–Poincaré symmetric, also enjoy some kind of symmetry under dilatations whose generator is the dilatation function. This is in accordance with a result by Zeeman [5] that the global symmetry transformations that leave causal relations invariant are uniform translations, homogeneous spacetime rotations, space inversion, and dilatations. Therefore, the minimal symmetry group is not 10 but 11 dimensional, with the generators P^μ, $J^{\mu\nu}$ and D. However, unlike P^μ and $J^{\mu\nu}$ and what is usually expected from global observables and generators of symmetries, it should also be noted that crucial in either of the definitions is that D *not be* a constant of the motion, since otherwise τ would have been a constant and could not serve as a time-like variable. This aspect is further discussed in the next section.

Finally, it is in place to point out that already Finkelstein, as early as 1949 [6], suggested expression (7) for the proper time in the context of quantum field theory (where X^μ is the position operator), but he insisted on keeping τ as a c-number parameter rather than a dynamical observable. His τ had then to be invariant under translations which clearly contradicts the required translational symmetry of X^μ (6) and τ (8). Johnson [7] considered a covariant position operator with the correct translational symmetry (6), but used the proper time τ only as an evolution parameter without relating it dynamically to the position operator. Kalnay and Cotrina [8] used equation (7) for the proper time in Dirac's theory of the electron following Finkelstein's proposal. They did regard it as an operator, but had already assumed several alternative expressions for the position operator that are relevant only in the context of Dirac's theory. In the context of classical (non-quantum) relativistic dynamics, the CM proper time τ was regarded just as a time-like parameter along the CM-coordinate spacetime trajectory, without paying attention to its properties and possible dynamic relations with the particles of the system (see, e.g., [9]). Apart from these attempts, no practical proposition seems to have been made since then that relates τ to dynamical properties of the particles. In particular, no relation has been made between the CM proper time or internal time and the dilatation function D.

4. The dilatation function for systems of interacting particles

To realize the foregoing results within a specific model, let us consider relativistic (non-quantum) many-particle systems which are symmetric under global Lorentz–Poincaré spacetime transformations. These could be systems of n point particles, with masses m_a and moving on the trajectories $x^\mu = x_a^\mu(\tau_a)$, $a = 1, \ldots, n$, with τ_a being the proper time of the ath particle. Being symmetric these systems are closed and endowed with conserved P^μ and $J^{\mu\nu}$.

Consider first a system of non-interacting particles with constant unit velocities u_a^μ. The dilatation function is simply given by

$$D = \sum_a x_a^\mu p_{a\mu} \tag{14}$$

with $p_{a\mu} = m_a u_{a\mu}$ the linear momenta of the single particles. Under uniform translations D does indeed transform as required by equation (10),

$$x_a^\mu \to x_a^\mu + a^\mu \quad \Rightarrow \quad D \to D + a^\mu P_\mu,$$

with total linear momentum $P_\mu = \sum_a p_{a\mu}$. The most important property of D for our purpose here is that it is *not* a constant of the motion—rather, it is a function of the proper times of all the particles with the differential

$$\mathrm{d}D(\{\tau_a\}) = \sum_a \mathrm{d}x_a^\mu p_{a\mu} = \sum_a u_a^\mu p_{a\mu}\, \mathrm{d}\tau_a = -\sum_a m_a\, \mathrm{d}\tau_a \tag{15}$$

so that the non-constancy depends on the masses of the particles.

This definition for D deviated from the usual trend, which looks for a *conserved* quantity which in our notation corresponds to $D_0 \equiv D + \sum_a m_a \tau_a$. We prefer here, and this preference is indeed crucial for the present work, the definition (14) for two reasons:

(i) τ, and thus also D, as observables, are expected to be functions of dynamical variables, namely spacetime coordinates and momenta of the particles. The particles' proper times are not dynamical variables.

(ii) The *non-constancy* of D is required for τ (in either of the above definitions) to be time variable.

Andersen and von Baeyer [10] extended the free-particle definition of D_0 based on (14) and have shown the existence of a (conserved, D_0-like) dilatation function for an electromagnetic two-body bound system with the Fokker–Wheeler–Feynman direct (action-at-a-distance) interaction [11, 12]. Their results were later generalized by Stephas and von Baeyer [13] for any number of particles interacting via the electromagnetic (vector) or scalar (EM-like) direct action. These results may be further generalized for a larger family of systems with direct interactions introducing a general interaction given by the action integrals, either for scalar interactions,

$$S = \sum_a - m_a \int \mathrm{d}\tau_a - \sum_{(a,b)} q_a q_b \iint U\left(x_{ab}{}^2\right) \mathrm{d}\tau_a \cdot \mathrm{d}\tau_b \tag{16a}$$

or for vector interactions,

$$S = \sum_a - m_a \int \mathrm{d}\tau_a + \sum_{(a,b)} q_a q_b \iint U\left(x_{ab}{}^2\right) \mathrm{d}x_a \cdot \mathrm{d}x_b \tag{16b}$$

where $\mathrm{d}\tau_a = \sqrt{-\mathrm{d}x_a \cdot \mathrm{d}x_a}$ and $x_{ab}{}^2 = (x_a - x_b)^2$, q_a is a generalized 'charge' and $U\left(x_{ab}{}^2\right)$ is a generalized kernel ($U(s) = \delta(s)$ in the electromagnetic case). It then follows (see the appendix for details) that a total dilatation function may be defined for both interactions as

$$D(\{\tau_a\}) \equiv \sum_a x_a^\mu(\tau_a) p_{a\mu}(\tau_a) - \sum_{(a,b)} q_a q_b \left(\int_{\tau_a}^{\infty} \int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a} \int_{\tau_b}^{\infty} \right) \times U'\left(x_{ab}{}^2\right)\left(x_a{}^2 - x_b{}^2\right)(\mathrm{d}\tau_a \cdot \mathrm{d}\tau_b) \qquad \text{(scalar interaction)} \tag{17a}$$

$$\equiv \sum_a x_a^\mu(\tau_a) p_{a\mu}(\tau_a) + \sum_{(a,b)} q_a q_b \left(\int_{\tau_a}^{\infty} \int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a} \int_{\tau_b}^{\infty} \right) \times U'\left(x_{ab}{}^2\right)\left(x_a{}^2 - x_b{}^2\right)(\mathrm{d}x_a \cdot \mathrm{d}x_b) \qquad \text{(vector interaction)} \tag{17b}$$

where the single particle's linear momenta are

$$p_{a\mu} = \frac{\delta S}{\delta x_a^\mu} = \left[m_a + q_a \sum_{b \neq a} q_b \int U\left(x_{ab}{}^2\right) \mathrm{d}\tau_b \right] \dot{x}_{a\mu} \qquad \text{(scalar interaction)} \tag{18a}$$

$$= m_a \dot{x}_{a\mu} + q_a \sum_{b \neq a} q_b \int U\left(x_{ab}{}^2\right) \mathrm{d}x_{b\mu} \qquad \text{(vector interaction)} \tag{18b}$$

with $\dot{x}_a^\mu = \mathrm{d}x_a^\mu / \mathrm{d}\tau$.

These dilatation functions are indeed Lorentz scalars, and under uniform translations they transform as required by equation (10). They are not conserved, and their differentials are

$$\mathrm{d}D(\{\tau_a\}) = -\sum_a \left\{ m_a + \sum_{b \neq a} q_a q_b \int_b \left[U\left(x_{ab}{}^2\right) + x_{ab}{}^2 U'\left(x_{ab}{}^2\right) \right] \mathrm{d}\tau_b \right\} \mathrm{d}\tau_a \qquad \text{(scalar interaction)} \tag{19a}$$

$$= -\sum_a \left\{ m_a - \sum_{b \neq a} q_a q_b \int_b \left[U\left(x_{ab}{}^2\right) + x_{ab}{}^2 U'\left(x_{ab}{}^2\right) \right] \dot{x}_a \cdot \mathrm{d}x_b \right\} \mathrm{d}\tau_a \qquad \text{(vector interaction).} \tag{19b}$$

For a general system of interacting particles, we apply the continuum picture and assume that the system is endowed with a conserved energy–momentum tensor, written as the sum of its kinetic (particles) and interaction parts

$$\begin{aligned} T^{\mu\nu}(x) &= T^{(\mathrm{par})\mu\nu}(x) + T^{(\mathrm{int})\mu\nu}(x) \\ &= \sum_a m_a \int_{-\infty}^{\infty} \delta^4[x - x_a(\tau_a)]\dot{x}_a^\mu \dot{x}_a^\nu \, \mathrm{d}\tau_a + T^{(\mathrm{int})\mu\nu}(x). \end{aligned} \tag{20}$$

Let Σ be an arbitrary space-like hypersurface. The trajectory of the ath particle cuts it at a single event identified by the proper time $\tau_a = \tau_a^\Sigma$, $x_a^\mu(\tau_a^\Sigma)$. If the interaction part depends only on the particles' trajectories then the dilatation function, defined as in relativistic field theory

$$D(\{\tau_a^\Sigma\}) = \int_\Sigma x^\mu T_\mu^\nu (\mathrm{d}^3 x)_\nu, \tag{21}$$

is a function of all the particles' proper times. D is not a constant, and to compute its variation let Σ' be another space-like hypersurface very close to Σ, so that $\tau_a^{\Sigma'} = \tau_a^\Sigma + \mathrm{d}\tau_a$, and let Ω be the spacetime domain between Σ and Σ'. Then we get

$$\mathrm{d}D(\{\tau_a^\Sigma\}) = \int_\Omega T_\mu^\mu (\mathrm{d}^4 x) = -\sum_a m_a \, \mathrm{d}\tau_a + \int_\Omega T^{(\mathrm{int})\mu}{}_\mu (\mathrm{d}^4 x). \tag{22}$$

In the following, where the notation is clear, we omit the superscript in τ_a^Σ.

The non-conservation of D is obviously a result of $T_\mu^\mu \neq 0$. It is interesting to note that for massless systems (such as pure EM radiation) where $T_\mu^\mu = 0$ and D is a constant, no rest frame and thus no proper time may be defined.

All the dilatation functions defined above are understood as functions of sets $\{\tau_a\}$ of proper times of all the particles, implicitly via their trajectories or momenta, so that to each arbitrary choice of $\{\tau_a\}$ or the corresponding events $\{x_a^\mu(\tau_a)\}$ along the particles' world lines there corresponds a value of D. In particular, if the interaction energy–momentum tensor in (22) is traceless,

$$T^{(\mathrm{int})\mu}{}_\mu = 0 \tag{23}$$

or the interaction kernel in (19*a*) and (19*b*) satisfies the homogeneity condition

$$U(x_{ab}{}^2) + x_{ab}{}^2 U'(x_{ab}{}^2) = 0 \tag{24}$$

corresponding to the propagator of the massless wave equation, then the differentials of D contain only single-particle kinetic terms and are identical in form with the non-interacting case (15).

Thus, for all the interactions that satisfy either equation (23) or (24) the differential of the corresponding dilatation function is independent of the interaction. In the following, we limit ourselves to these cases (which include anyway the electromagnetic interaction) and shall not deal with other forms of interactions (see the appendix for possible extensions).

We therefore consider all the dilatation functions that share the same differential property

$$\mathrm{d}D(\{\tau_a\}) = -\sum_a m_a \, \mathrm{d}\tau_a \tag{25}$$

which integrates to

$$D(\{\tau_a\}) = D_0 - \sum_a m_a \tau_a \tag{26}$$

independently of the details of the particles' trajectories (except, possibly, in D_0). It is this expression for D that we insert in equation (11) or (13) for the proper-time observable τ.

It should be noted that the combination $D_0 = D(\{\tau_a\}) + \sum_a m_a \tau_a$ is a constant of the motion, but it cannot be regarded as an observable because of its explicit dependence on the proper times of the particles. (The double integral interaction term in either (17*a*) or (17*b*) depends on the τ via the limits of the integrals, but still it does so as a functional of the trajectories of the particles. The sum $\sum_a m_a \tau_a$, on the other hand, depends explicitly only on the τ, without any 'behind the scenes' dependence on the particles' trajectories, and thus cannot be regarded as an observable.)

Finally, it is also noted that the homogeneity condition (24) implies that the actions (16*a*) and (16*b*) are invariant under uniform scaling of the coordinates and particles' parameters in the form

$$x^\mu \to \mathrm{e}^\lambda x^\mu, \qquad m_a \to \mathrm{e}^{-\lambda} m_a, \qquad q_a \to q_a \tag{27}$$

thus confirming that D is indeed associated with arbitrary uniform scalings. Under these scalings the system's observables transform as expected,

$$P^\mu \to \mathrm{e}^{-\lambda} P^\mu, \qquad J^{\mu\nu} \to J^{\mu\nu}, \qquad D \to D,$$

from which it is verified that X^μ also scales as a coordinate,

$$X^\mu \to \mathrm{e}^\lambda X^\mu.$$

5. Simultaneity in the CM reference frame

We now use this many-particle system model to examine the two alternatives (11) and (13) proposed for the proper-time observable τ.

Let us pick up arbitrarily an event on each particle's trajectory, defining together an arbitrary set of particles' events $\left\{x_a^\mu(\tau_a)\right\}$ with the corresponding set of proper times $\{\tau_a\}$. Each such set defines, via equation (11) or (13), a unique value of the CM proper time $\tau(\{\tau_a\})$ as a function of the proper times of the particles (figure 1). In particular, all the particle's events may be chosen so that they are simultaneous in the CM reference frame (figure 2). Then the question arises: Are these events also simultaneous with the CM proper time that they define? In other words, Are the events that constitute τ as internal time simultaneous, in the CM reference frame, with the time τ as measured along the CM-time axis?

This simultaneity question stems, of course, from the fact that τ has here a double role: being, on the one hand, an observable whose value $\tau(\{\tau_a\})$ is defined by the arbitrarily chosen set of events $\left\{x_a^\mu(\tau_a)\right\}$, but, on the other hand, being the proper time measured along the CM world line. Naive intuition directs us to expect identity of these two roles; however, as becomes evident in the following, it turns out that the identity is not automatically ensured.

We start with the CM proper-time observable as defined by equation (11), identifying $X \cdot P$ with D. Substituting equation (26) it is given by

$$\tau(\{\tau_a\}) = -\frac{D(\{\tau_a\})}{M} = \frac{\sum_a m_a \tau_a - D_0}{M}. \tag{28}$$

Consider the hyperplane $x^0 = t$ in the CM reference frame. Each particle's trajectory (given here as observed in the CM frame) $x^\mu = x_a^\mu(\tau_a)$ cuts this hyperplane only once, defining for each τ_a a unique function $\tau_a^{\mathrm{CM}}(t)$ via the identity

$$x_a^0(\tau_a) = t \quad \Leftrightarrow \quad \tau_a = \tau_a^{\mathrm{CM}}(t). \tag{29}$$

All the particles' events that correspond to these $\tau_a = \tau_a^{\mathrm{CM}}(t)$ are simultaneous with the CM-observer time t, as in figure 2. Using these values of τ_a, one may compute via equation (28) the corresponding proper-time observable

$$\tau(t) = \frac{\sum_a m_a \tau_a^{\mathrm{CM}}(t) - D_0}{M}. \tag{30}$$

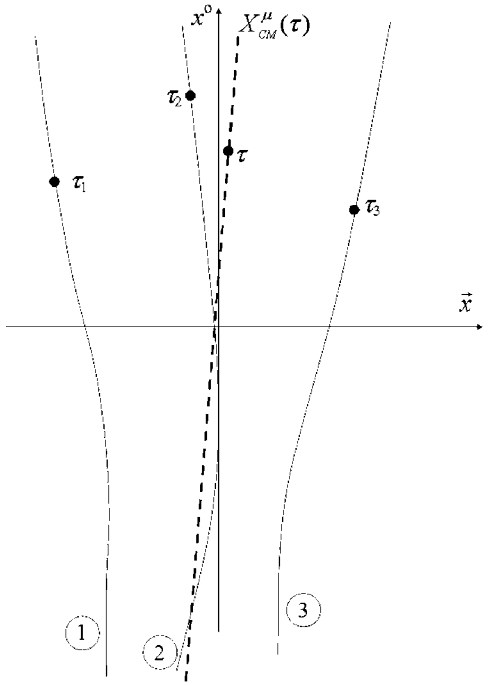

Figure 1. The proper time $\tau(\{\tau_a\})$ corresponding to a set of events $\{x_a^{\mu}(\tau_a)\}$ of a system of particles in a general reference frame.

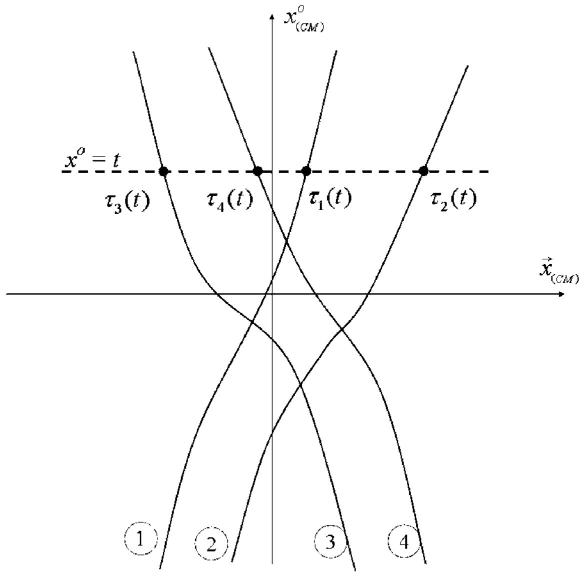

Figure 2. The simultaneity problem for a system of particles in the CM frame.

The particles' events corresponding to $\left\{\tau_a = \tau_a^{\text{CM}}(t)\right\}$ are simultaneous with the corresponding proper-time observable if and only if $\tau(t) = t$, at least up to an additive constant. Thus, the simultaneity question may be rephrased as follows: Does $\tau(t)$ coincide with t, for every conceivable system of particles?

To answer the question, differentiating equation (30) with respect to t (taking into account the constancy of D_0) yields

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{M}\sum_a m_a \frac{\mathrm{d}\tau_a^{\mathrm{CM}}(t)}{\mathrm{d}t} = \frac{1}{M}\sum_a m_a \gamma^{-1}(v_a) \tag{31}$$

where v_a is the ath particle's velocity as measured in the CM reference frame and the relation $\mathrm{d}\tau_a^{\mathrm{CM}}(t) = \gamma^{-1}(v_a)\,\mathrm{d}t = \sqrt{1-v_a^2}\,\mathrm{d}t$ is used. Since proper times and the time coordinate may be defined up to an additive constant, it follows from equation (31) that a necessary and sufficient condition for the system's proper time to be simultaneous with the events that determine it is that the rhs of equation (31) be equal to 1, namely the existence of the relation

$$M = \sum_a m_a {\gamma_a}^{-1} \tag{32}$$

in the CM reference frame.

Relation (32) puts a very stringent condition on the system. It was shown by Schild [9] for a two-body electromagnetic system, and recently extended by Louis–Martinez [14] for an arbitrary number of particles, that this relation exists in the case of circular motion of all the particles around their common centre-of-mass. It is not satisfied in any other solution known today: in the case of general bound systems relation (32) exists only on the average [10],

$$M = \sum_a m_a \langle {\gamma_a}^{-1} \rangle \tag{33}$$

and in the case of unbound systems it seems to be always violated, since in particular for a system of free particles the total mass is

$$M = \sum_a m_a \gamma_a.$$

As a simple illustration of this issue of simultaneity let us consider a system of n non-interacting particles moving on the trajectories $x_a^\mu(\tau_a) = u_a^\mu \tau_a + x_{a0}^\mu$ with normalized 4-velocities u_a^μ. With the particles' proper times on a common CM-time hyperplane $-x^\mu U_\mu = t$ given by (29)

$$\tau_a^{\mathrm{CM}}(t) = \frac{t + x_{a0}\cdot U}{-u_a\cdot U}$$

the dilatation function becomes

$$D = \sum_a m_a x_a\cdot u_a = -\sum_a m_a\tau_a + \sum_a m_a x_{a0}\cdot u_a$$
$$= \sum_a \frac{m_a t}{u_a\cdot U} + \sum_a m_a\left[\frac{x_{a0}\cdot U}{u_a\cdot U} + x_{a0}\cdot u_a\right]$$

so that the corresponding proper-time observable equation (30) is

$$\tau(t) = -\frac{D}{M} = \left(\sum_a \frac{m_a}{-u_a\cdot P}\right)\cdot t - \sum_a \frac{m_a}{M}\left[\frac{x_{a0}\cdot U}{u_a\cdot U} + x_{a0}\cdot u_a\right].$$

Clearly, the coefficient of t on the rhs, equal to $\sum_a (m_a/M)\gamma_a^{-1}(v_a)$, is less than 1, demonstrating indeed that $\tau(t)$ cannot coincide with t itself.

We may therefore conclude that with the identification $X\cdot P = D$, with the corresponding definition of τ by equation (11), it is impossible, in general, to describe the particles' system simultaneously in a time that is common to all the particles and coincides with the CM proper time.

6. Solution of the simultaneity problem

A necessary outcome of the results of the previous section is that if it is insisted that $X \cdot P = D$ (say, as part of a requirement that X^μ be defined only from P^μ, $J^{\mu\nu}$ and D) so that equation (11) holds then the concept of simultaneity must be modified. It is necessary then to consider an extended sense of simultaneity, which is accomplished on curved hypersurfaces rather than hyperplanes (for instance, in the case of non-interacting particles these hypersurfaces are hyperboloids symmetric around the CM-time axis).

However, the inability of the CM proper time thus defined to be simultaneous (in the common sense) with the events that define it seems to be in disaccord with naive intuition, which would expect such a simultaneity to always occur. Although this new sense of simultaneity could be a valid result and our intuition wrong, it looks more like an anomaly, and we would be more at ease if this anomaly could be removed. It will be shown now that it is possible to retain the concept of simultaneity in its common sense by splitting D according to equation (12) and using the corresponding internal-time definition (13).

Let us first consider a pair of non-interacting particles, moving on the trajectories $x_a^\mu(\tau_a) = u_a^\mu \tau_a + x_{a0}^\mu$ with normalized 4-velocities u_a^μ $(a = 1, 2)$. The action differential may be decomposed to a combination of a global part and an internal (relative to the CM frame) part, up to a total differential,

$$\begin{aligned} \mathrm{d}S &= p_1 \cdot \mathrm{d}x_1 + p_2 \cdot \mathrm{d}x_2 \\ &= P \cdot \mathrm{d}X + q \cdot \mathrm{d}\xi + d\left[\left(m_1^2 - m_2^2\right)\frac{P \cdot (x_1 - x_2)}{M^2}\right] \end{aligned} \tag{34}$$

where

$$q^\mu = p_1^\mu + (p_1 \cdot U)U^\mu = -p_2^\mu - (p_2 \cdot U)U^\mu \tag{35}$$

is the single particle's momentum relative to the CM frame ($q \cdot P = 0$),

$$\xi^\mu = x_1^\mu - x_2^\mu + \frac{P \cdot (x_1 - x_2)}{M^2} P^\mu$$

is the relative spatial vector in the CM frame, and X^μ suggests an expression for the CM coordinate,

$$X^\mu = -\frac{p_1 \cdot P}{M^2} x_1^\mu - \frac{p_2 \cdot P}{M^2} x_2^\mu + \frac{(x_1 - x_2) \cdot P}{M^2}\left(\frac{m_1^2 - m_2^2}{M^2} P^\mu + q^\mu\right). \tag{36}$$

The product $X \cdot P$ is clearly different from D, the difference being

$$\begin{aligned} D_{\mathrm{int}} &= D - X \cdot P \\ &= x_1 \cdot p_1 + x_2 \cdot p_2 + \frac{p_1 \cdot P}{M^2} x_1 \cdot P + \frac{p_2 \cdot P}{M^2} x_2 \cdot P + \left(\frac{m_1^2 - m_2^2}{M^2}\right)(x_1 - x_2) \cdot P \\ &= (x_1 - x_2) \cdot q + \left(\frac{m_1^2 - m_2^2}{M^2}\right)(x_1 - x_2) \cdot P. \end{aligned} \tag{37}$$

The pair's common internal time then follows, with some algebra and using the single particles' trajectories explicitly, as

$$\begin{aligned} \tau &= -\frac{X \cdot P}{M} = \frac{p_2 \cdot P}{M^3} x_1 \cdot P + \frac{p_1 \cdot P}{M^3} x_2 \cdot P \\ &= -\frac{D_0}{M} + \frac{(p_1 \cdot P)(p_2 \cdot P)}{M^3}\left(\frac{\tau_1}{m_1} + \frac{\tau_2}{m_2}\right). \end{aligned} \tag{38}$$

(For convenience, the origins of τ_1 and τ_2 are fixed by the conditions $(x_{1o} - x_{2o}) \cdot u_1 = (x_{1o} - x_{2o}) \cdot u_2 = 0$.) These are the final expressions for τ both as an observable and as an explicit function of the single particles' proper times. It is interesting to note, in passing, that the relative spatial vector in the CM frame, expressed in terms of the single particles' proper times,

$$\begin{aligned} \xi^\mu(\tau_1, \tau_2) &= x_1^\mu(\tau_1) - x_2^\mu(\tau_2) + \frac{[x_1(\tau_1) - x_2(\tau_2)] \cdot P}{M^2} P^\mu \\ &= x_{1o}^\mu - x_{2o}^\mu + q^\mu \left(\frac{\tau_1}{m_1} + \frac{\tau_2}{m_2} \right), \end{aligned} \tag{39}$$

is fully expressible as a function of τ alone, for any choice of τ_1 and τ_2.

Next we consider a system of n non-interacting particles with unit 4-velocities u_a^μ. The single particle's momentum relative to the CM frame is the component of p_a^μ orthogonal to P^μ, $q_a^\mu = p_a^\mu + (p_a \cdot U)U^\mu$. On a common CM-time hyperplane $-x^\mu U_\mu = t$ the dilatation function D is

$$\begin{aligned} D(t) &= \sum_a x_a \cdot p_a = \sum_a -(p_a \cdot U)(x_a \cdot U) + \sum_a x_a \cdot q_a \\ &= -\sum_a m_a \gamma(v_a) x_a^0 + \sum_a x_a \cdot q_a = -Mt + \sum_a x_a \cdot q_a. \end{aligned}$$

In accordance with equation (37), the common CM-time internal part of the dilatation function is identified as

$$D_{\rm int}(t) \equiv \sum_a x_a \cdot q_a \tag{40}$$

from which follows indeed the relation

$$t = -\frac{D - D_{\rm int}}{M} \tag{41}$$

expressing the equality of the common CM time to the internal time defined by equation (13).

For a general interacting system, using the continuum picture, let the CM-time hyperplane $-x^\mu U_\mu = t$ be the integration surface Σ in equation (21). Then we obtain

$$D(t) = \int_{x^0=t} x^\mu T_\mu^0 \, {\rm d}V = t \int_{x^0=t} T_0^0 \, {\rm d}V + \int_{x^0=t} x^i T_i^0 \, {\rm d}V = -Mt + \int_{x^0=t} x^i T_i^0 \, {\rm d}V \tag{42}$$

so that identifying

$$D_{\rm int}(t) \equiv \int_{x^0=t} x^i T_i^0 \, {\rm d}V \tag{43}$$

equation (41) is recovered.

In conclusion, it seems more natural and reasonable to accept equation (13) as the definition of the internal-time observable. Although a more general expression for the internal dilatation function $D_{\rm int} = D - X^\mu P_\mu$, suitable for arbitrary collection of events with proper times $\{\tau_1, \ldots, \tau_n\}$ or integration hypersurface Σ in equation (21), is not available at the moment, nevertheless it can be seen to provide a measure of the changes that the internal configuration of the system undergoes during its evolution.

The validity of relation (32) for many-particle systems in circular motion implies that the equal-time $D_{\rm int}$, computed on a common CM-time hyperplane $t = -x \cdot U$, vanishes in these cases. In the continuum case, $D_{\rm int}(t)$ vanishes for systems with circular symmetry because T_i^0 must be angularly directed relative to the centre-of-mass. Thus, we conclude that circular symmetry implies $D_{\rm int}(t) = 0$. It may be noted that equation (37) suggests that the vanishing of D_{int} in the case of circular symmetry is limited, in general, only for sets of events $\{x_a^\mu(\tau_a)\}$ which are CM-time simultaneous. For general bound systems $D_{\mathrm{int}}(t)$ is non-zero but bounded with zero average, and application of this fact leads ([13, 16]) to the relativistic virial theorem for electromagnetic systems, contained in equation (33). In the general case it bears the very peculiar property of being time dependent, while being invariant under the global symmetries of uniform spacetime translations, rotations and scaling; it may thus be regarded as being 'transparent' for these symmetry transformations.

7. Discussion and concluding remarks

We have shown how the internal time of classical (non-quantum) relativistic systems, identical with the system's proper time, is naturally defined as a Lorentz-invariant observable simply related to the dilatation function D of the system, the latter being split into CM and internal parts. The system is therefore required to be symmetric not only under Lorentz–Poincaré transformations but also under uniform scaling.

A major novelty here is in the fact that this definition requires D to be a varying—rather than conserved—quantity. It thus turns out that being a varying quantity is a necessary requisite for the dilatation function to be a main tool in measuring the aging and internal evolution of the system.

Still, there are those many-particle systems (including, in particular, electromagnetically interacting) for which the variation of D depends only upon the particles' proper times and is independent of the details of the interactions. This property allowed us to discuss the internal time of these systems uniformly, without recourse to the details of the interactions.

In deriving our results, we made use of the dual role that time has in Nature and is disclosed in relativistic dynamics: time as measured by the clocks of any reference frame is external to physical systems and cannot be an observable; but the actual measurement of time with respect to any particular system is local, depending on its own proper time. It is only on this conceptual basis of relativistic dynamics that we can talk of the internal time of a physical system as an observable that can be a measure of its internal evolution[1].

It is of interest to note how the internal-time scheme presented here stands in relation to other internal-time or relativistic CM schemes; in particular, regarding the association between dilatations and the proper or internal time of the particles' system. All the other schemes for relativistic CM coordinate and/or internal time seem to insist, at least explicitly, on defining the internal time within the framework of the Lorentz–Poincaré symmetry only (see, e.g., [17, 18] and references therein, or [19]); this, in spite of the fact that the corresponding Lorentz–Poincaré generators are constants of the motion and cannot provide for a time-like varying observable. Apparently, none of these other schemes seeks to extend beyond the Lorentz–Poincaré symmetry, and therefore none of them has reached the point of associating the internal-time observable with the dilatation function.

However, work done with the Newtonian definitions of internal time shows some relations with dilatations: the (non-relativistic) internal-time operator T of Misra, Prigogine and Courbage [1] is associated with what are known in ergodic theory as K-flows, which are characterized by exponentially diverging trajectories. This is to be compared with the fact that the integral curves of the dilatation function D also have the nature of exponential growth, $x^\mu(\lambda) = \mathrm{e}^\lambda x^\mu(0)$. A similar relation may be found in the discussion by Lockhart, Misra and Prigogine [20] of geodesic flow in Friedman Universes with negative curvature which

[1] For a recent similar emphasis on the distinction between the observers' time and the proper time, but from a different approach, see [15].

also diverges exponentially. Also in the examples given by Vallèe [2] there is exponential divergence of the trajectories associated with the definition of the internal time. Finally, it is of interest to note the fact, hitherto not realized, that the internal time T of Misra, Prigogine and Courbage extends naturally into the internal-time scheme presented here: being defined (for Hamiltonian systems) via its commutation relation with the Hamiltonian $[H, T] = \imath$, it can be shown that combining this commutation relation (or the corresponding Poisson brackets for classical systems) with the Lorentz–Poincaré symmetry leads naturally to the existence of a dilatation function D and consequently to the inclusion of scaling symmetry.

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Appendix. The dilatation function in action-at-a-distance theories

This appendix is devoted to showing the existence of a dilatation function for the systems of relativistic interacting particles presented in section 4. Varying the action integrals (16*a*) and (16*b*) yields the single particle's linear momenta for both the scalar and vector interactions, given in equations (18*a*) and (18*b*), and their equations of motion

$$\frac{\mathrm{d}p_a^\mu}{\mathrm{d}\tau_a} = -2q_a \sum_{b\neq a} q_b \int U'\left({x_{ab}}^2\right)\left(x_a^\mu - x_b^\mu\right)\mathrm{d}\tau_b \qquad \text{(scalar interaction)}$$

$$= 2q_a \sum_{b\neq a} q_b \int_b U'\left({x_{ab}}^2\right)\left(x_a^\mu - x_b^\mu\right)(\dot{x}_a \cdot \mathrm{d}x_b) \qquad \text{(vector interaction)}.$$

The total linear momenta

$$P^\mu(\{\tau_a\}) \equiv \sum_a p_a^\mu(\tau_a) - \sum_{(a,b)} 2q_a q_b \left(\int_{\tau_a}^{\infty}\int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a}\int_{\tau_b}^{\infty}\right)$$
$$\times U'\left({x_{ab}}^2\right)\left(x_a^\mu - x_b^\mu\right)(\mathrm{d}\tau_a \cdot \mathrm{d}\tau_b) \qquad \text{(scalar interaction)}$$
$$\equiv \sum_a p_a^\mu(\tau_a) + \sum_{(a,b)} 2q_a q_b \left(\int_{\tau_a}^{\infty}\int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a}\int_{\tau_b}^{\infty}\right)$$
$$\times U'\left({x_{ab}}^2\right)\left(x_a^\mu - x_b^\mu\right)(\mathrm{d}x_a \cdot \mathrm{d}x_b) \qquad \text{(vector interaction)}$$

and total angular momenta

$$J^{\mu\nu}(\{\tau_a\}) \equiv \sum_a \left[x_a^\mu(\tau_a)p_a^\nu(\tau_a) - x_a^\nu(\tau_a)p_a^\mu(\tau_a)\right] + \sum_{(a,b)} 2q_a q_b \left(\int_{\tau_a}^{\infty}\int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a}\int_{\tau_b}^{\infty}\right)$$
$$\times U'\left({x_{ab}}^2\right)\left(x_a^\mu x_b^\nu - x_a^\nu x_b^\mu\right)(\mathrm{d}\tau_a \cdot \mathrm{d}\tau_b) \qquad \text{(scalar interaction)}$$
$$\equiv \sum_a \left[x_a^\mu(\tau_a)p_a^\nu(\tau_a) - x_a^\nu(\tau_a)p_a^\mu(\tau_a)\right]$$
$$+ \sum_{(a,b)} q_a q_b \left(\int_{\tau_a}^{\infty}\int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a}\int_{\tau_b}^{\infty}\right) U\left({x_{ab}}^2\right)\left(\mathrm{d}x_a^\mu\,\mathrm{d}x_b^\nu - \mathrm{d}x_a^\nu\,\mathrm{d}x_b^\mu\right)$$

$$+\sum_{(a,b)} 2q_a q_b \left(\int_{\tau_a}^{\infty} \int_{-\infty}^{\tau_b} - \int_{-\infty}^{\tau_a} \int_{\tau_b}^{\infty} \right)$$
$$\times U'\left(x_{ab}{}^2\right)\left(x_a^\nu x_b^\mu - x_a^\mu x_b^\nu\right)(\mathrm{d}x_a \cdot \mathrm{d}x_b) \qquad \text{(vector interaction)}$$

are conserved, their conservation (for both interactions) given by the relations

$$\frac{\partial P^\mu}{\partial \tau_a} = 0, \qquad \frac{\partial J^{\mu\nu}}{\partial \tau_a} = 0.$$

For the definition of the total dilatation function, we first compute the differentials

$$\mathrm{d}\left(\sum_a x_a^\mu p_{a\mu}\right) = -\sum_a m_a \,\mathrm{d}\tau_a - \sum_a \sum_{b\neq a} q_a q_b \int_b \left\{U\left(x_{ab}{}^2\right)\right.$$
$$\left. + 2U'\left(x_{ab}{}^2\right)[x_a \cdot (x_a - x_b)]\right\} \mathrm{d}\tau_b \,\mathrm{d}\tau_a \qquad \text{(scalar interaction)}$$
$$= -\sum_a m_a \,\mathrm{d}\tau_a + \sum_a \sum_{b\neq a} q_a q_b \int_b \left\{U\left(x_{ab}{}^2\right)\right.$$
$$\left. + 2U'\left(x_{ab}{}^2\right)[x_a \cdot (x_a - x_b)]\right\} \mathrm{d}x_a \cdot \mathrm{d}x_b \qquad \text{(vector interaction).}$$

The integrand in both cases may be reordered, using

$$2x_a = (x_a - x_b) + (x_a + x_b)$$

as

$$U\left(x_{ab}{}^2\right) + 2U'\left(x_{ab}{}^2\right)[x_a \cdot (x_a - x_b)] = U\left(x_{ab}{}^2\right) + x_{ab}{}^2 U'\left(x_{ab}{}^2\right) + U'\left(x_{ab}{}^2\right)\left(x_a{}^2 - x_b{}^2\right).$$

The last term in the rhs, which is anti-symmetric in a and b, is used to define a double integral interaction term in a manner similar to that in P^μ and $J^{\mu\nu}$. The other two terms, which are symmetric in a and b, cannot define (due to their symmetry) an interaction term in a similar way, and we are thus led to defining the total dilatation function as in equations (17a) and (17b) with differentials given by equations (19a) and (19b), correspondingly.

These dilatation functions reduce to the results of Stephas and von Baeyer [13] for the electromagnetic (or more general, massless field) case, with $U\left(x_{ab}{}^2\right) = \delta\left(x_{ab}{}^2\right)$ and the scaling scheme (27). A more general scaling scheme, in which the charges also scale along with the coordinates and the masses, is possible with the recipe

$$x^\mu \to \mathrm{e}^\lambda x^\mu, \qquad m_a \to \mathrm{e}^{-\lambda} m_a, \qquad q_a \to \mathrm{e}^{-n\lambda} q_a$$

with n being an integer. The action integrals are then scale invariant and the observables of the system transform as expected under scaling,

$$P^\mu \to \mathrm{e}^{-\lambda} P^\mu, \qquad J^{\mu\nu} \to J^{\mu\nu}, \qquad D \to D,$$

if $U(s)$ is homogeneous of order $n-1$ in its argument. Using the homogeneity condition

$$sU'(s) = (n-1)U(s)$$

it follows that the differentials of D are

$$\mathrm{d}D(\{\tau_a\}) = -\sum_a \left[m_a + \sum_{b\neq a} n q_a q_b \int_b U\left(x_{ab}{}^2\right) \mathrm{d}\tau_b\right] \mathrm{d}\tau_a \qquad \text{(scalar interaction)}$$
$$= -\sum_a \left[m_a - \sum_{b\neq a} n q_a q_b \int_b U\left(x_{ab}{}^2\right) \dot{x}_a \cdot \mathrm{d}x_b\right] \mathrm{d}\tau_a \qquad \text{(vector interaction).}$$

Unlike the massless field ($n = 0$) case, these differentials depend explicitly on the interaction.

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